A GENERALIZED PÓLYA'S URN WITH GRAPH BASED INTERACTIONS

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Abstract. Given a finite connected graph G, place a bin at each vertex. Two bins are called a pair if they share an edge of G. At discrete times, a ball is added to each pair of bins. In a pair of bins, one of the bins gets the ball with probability proportional to its current number of balls raised by some fixed power $\alpha \geq 0$. We characterize the limiting behavior of the proportion of balls in the bins when G is regular and $\alpha \leq 1$. For example, when G is non-bipartite the proportion converges to the uniform measure almost surely. A novel part of the proof consists of analyzing the equilibrium set of a non uniformly hyperbolic dynamical system.

1. Introduction and statement of results

Let $G = (V, E)$ be a finite connected graph with $V = \{1, \ldots, m\}$ and $|E| = N$, and assume that on each vertex i there is a bin initially with $B_i(0) \geq 1$ balls. For a fixed parameter $\alpha \geq 0$, consider a random process of adding N balls to these bins at each step, according to the following evolution law: if the numbers of balls after step $n-1$ are $B_1(n-1), \ldots, B_m(n-1)$, step n consists of adding, to each edge $\{i, j\} \in E$, one ball either to i or j, and the probability that the ball is added to i is

$$\mathbb{P}\left[i \text{ is chosen among } \{i,j\} \text{ at step } n\right] = \frac{B_i(n-1)^\alpha}{B_i(n-1)^\alpha + B_j(n-1)^\alpha} \cdot \qquad (1.1)$$

The number of balls at vertex i after step n is thus $B_i(n) = B_i(n-1) + C_i(n)$, where $C_i(n)$ is the number of balls added to i at step n.

In this paper, we study the limiting behavior, as the number of steps grows, of the proportion of balls on the vertices of G. More specifically, let $N_0 = \sum_{i=1}^m B_i(0)$ denote the initial total number of balls, let

$$x_i(n) = \frac{B_i(n)}{N_0 + nN}, \quad i = 1, \ldots, m, \qquad (1.2)$$

be the proportion of balls at vertex i after step n, and let $x(n) = (x_1(n), \ldots, x_m(n))$. Call the point $(1/m, \ldots, 1/m)$ the *uniform measure*. The first result classifies the limiting behavior of $x(n)$ when G is regular and $\alpha = 1$.

Theorem 1.1. *Let G be a finite, regular, connected graph, and let $\alpha = 1$.*

(a) If G is non-bipartite, then $x(n)$ converges to the uniform measure almost surely.
(b) If G is bipartite, then $x(n)$ converges to $\overline{\Lambda}$ almost surely.

Date: November 22, 2012.
2010 *Mathematics Subject Classification.* Primary: 60K35. Secondary: 37C10.
Key words and phrases. Chain-recurrence, ordinary differential equations, Pólya's urn, stochastic approximations.

Above, Λ is a subset of the $(m-1)$-dimensional open simplex Δ defined as follows: if $V = A \cup B$ is the bipartition of G, then

$$\Lambda = \{(x_1, \ldots, x_m) \in \Delta : \exists p, q > 0 \text{ s.t. } x_i = p \text{ on } A \text{ and } x_i = q \text{ on } B\}.$$

Theorem 1.1 includes the case of any finite complete graph. A complete graph with at least three vertices is non-bipartite and then the random process converges to the uniform measure almost surely.

Theorem 1.1 also includes the case of cycles: if the length is odd, the random process converges almost surely to the uniform measure; if the length is even, the random process' limit set is contained in $\overline{\Lambda}$ almost surely.

The next result shows that, when G is regular and $\alpha < 1$, $x(n)$ converges to the uniform measure, regardless of G being bipartite or not.

Theorem 1.2. *Let G be a finite, regular, connected graph, and let $\alpha < 1$. Then $x(n)$ converges to the uniform measure almost surely.*

The last main result deals with star graphs. The *star graph* is a tree with m vertices and $m-1$ leaves.

Theorem 1.3. *Let G be a finite star graph with at least three vertices, and let m be the vertex of highest degree.*
(a) If $\alpha = 1$, then $x(n)$ converges to $(0, \ldots, 0, 1)$ almost surely.
(b) If $\alpha > 1$, then with positive probability $x(n)$ converges to either

$$(0, \ldots, 0, 1) \quad \textit{or} \quad \left(\frac{1}{m-1}, \ldots, \frac{1}{m-1}, 0\right).$$

Imagine there are 3 companies, denoted by M, A and G. Each company sells two products. M sells OS and SE, A sells OS and SP, G sells SE and SP. Each pair of companies compete on one product. The companies try to use their global size and reputation to boost sales. A natural question is which company will sell more products in the long term. In this scenario, one can easily see that the interacting relation among the three companies forms a triangular network. On this triangle, a vertex represents a company and an edge represents a product. Under further simplifications, the model proposed above describes in broad strokes the long term evolution of such a competition.

Another motivation for the model comes from a repeated game in which agents improve their skills by gaining experience. The interaction network between agents is modeled by a graph. At each round a pair is competing for a ball. A competitor improves its skills with time, which is represented by the number of balls in its bin.

There are several natural ways to generalize our model in order to capture more complex interactions, but we focus here on the simplest setup. The model can be viewed as a class of graph based evolutionary model, which has been studied in various fields, e.g. biology [11], economics [5], mathematics [15], and sociology [14].

The classical *Pólya's urn* contains balls of two colors, and at each step one ball is drawn randomly, its color is observed, and a ball of the same color is added to the urn. In our process the interactions occur among pairs of bins, and on each of them the model evolves similar to the classical Pólya's urn. We therefore call it a *generalized Pólya's urn with graph based interactions*. For other variations on Pólya's urn, see e.g. [2, 8, 10].

We use a dynamical approach to prove Theorems 1.1 and 1.2. Benaïm [1] proved that in many circumstances the limit set of a random process is contained in the

chain-recurrent set of an associated semiflow. The novelty of our work is the proof of Theorem 1.1 when G is bipartite. The chain-recurrent set of the semiflow is Λ. Unlike in classical situations, Λ is not hyperbolic: although each point of Λ is a stable singularity, they lose hyperbolicity as they approach the extremes of Λ, and in the limit they degenerate to non-hyperbolic points. To overcome this difficulty, we extend the semiflow to neighborhoods of the extremes of Λ and construct Lyapunov functions on these neighborhoods that clarify the local behavior of the extended semiflow.

The paper is organized as follows. In Section 2 we make a brief discussion on stochastic approximation algorithms. In Section 3 we introduce the dynamical approach. In Sections 4, 5 and 6 we prove Theorems 1.1, 1.2 and 1.3, respectively. In Section 7 we discuss some variants of the model. Finally, we collect some questions in Section 8.

2. Stochastic approximation algorithms

A *stochastic approximation algorithm* is a discrete time stochastic process whose general form can be written as

$$x(n+1) - x(n) = \gamma_n H(x(n), \xi(n)) \tag{2.1}$$

where $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ is a measurable function that characterizes the algorithm, $\{x(n)\}_{n\geq 0} \subset \mathbb{R}^m$ is the sequence of parameters to be recursively updated, $\{\xi(n)\}_{n\geq 0} \subset \mathbb{R}^m$ is a sequence of random inputs where $H(x(n), \xi(n))$ is observable, and $\{\gamma_n\}_{n\geq 0}$ is a sequence of "small" nonnegative scalar gains. Such processes were first introduced in the early 50s on the works of Robbins and Monro [13] and Kiefer and Wolfowitz [6].

We now show that the random process defined by (1.2) is a stochastic approximation algorithm. We start by specifically formulating the model. Write $i \sim j$ whenever $\{i, j\} \in E$. Let $\mathcal{F}_n$ denote the sigma-algebra of the process up to step n. For each neighboring pair of vertices $i \sim j$, consider 0-1 valued random variables $\delta_{i\leftarrow j}(n+1), \delta_{j\leftarrow i}(n+1)$ such that $\delta_{i\leftarrow j}(n+1) + \delta_{j\leftarrow i}(n+1) = 1$ and

$$\mathbb{E}\left[\delta_{i\leftarrow j}(n+1)|\mathcal{F}_n\right] = \frac{B_i(n)^\alpha}{B_i(n)^\alpha + B_j(n)^\alpha} = \frac{x_i(n)^\alpha}{x_i(n)^\alpha + x_j(n)^\alpha}\,\cdot \tag{2.2}$$

Also, assume that $\delta_{i\leftarrow j}(n+1)$ and $\delta_{i'\leftarrow j'}(n+1)$ are independent whenever the edges $\{i, j\}$ and $\{i', j'\}$ are distinct. Thus

$$C_i(n+1) = \sum_{j\sim i} \delta_{i\leftarrow j}(n+1). \tag{2.3}$$

We want to show that $x(n) = (x_1(n), \ldots, x_m(n))$ satisfies a difference equation (2.1). Observe that

$$\begin{aligned} x_i(n+1) - x_i(n) &= \frac{B_i(n) + C_i(n+1)}{N_0 + (n+1)N} - \frac{B_i(n)}{N_0 + nN} \\ &= \frac{-Nx_i(n) + C_i(n+1)}{N_0 + (n+1)N} \\ &= \frac{1}{\frac{N_0}{N} + (n+1)}\left(-x_i(n) + \frac{1}{N}C_i(n+1)\right) \end{aligned}$$

and so $x(n)$ satisfies (2.1) with

$$\gamma_n = \frac{1}{\frac{N_0}{N} + (n+1)} \quad , \quad \xi(n) = \frac{1}{N}\left(C_1(n+1), \dots, C_m(n+1)\right) \tag{2.4}$$

and $H : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$H(x(n), \xi(n)) = -x(n) + \xi(n).$$

Conditioning on $\mathcal{F}_n$, H has a deterministic component $x(n)$ and a random component $\xi(n)$. Nevertheless, nothing can be said about converging properties of $\xi(n)$. To this matter, we modify the above equation by decoupling $\xi(n)$ into its mean part and the so called "noise" part, which has zero mean. If one manages to control the total error of the noise term, the limiting behavior of $x(n)$ can be identified via the limiting behavior of the new deterministic component.

3. The dynamical approach

The dynamical approach is a method introduced by Ljung [9] and Kushner and Clark [7] to analyze stochastic approximation algorithms. Formally, it says that recursive expressions of the form (2.1) can be analyzed via an averaged ordinary differential equation (ODE)

$$\frac{dx(t)}{dt} = \overline{H}(x(t))\,, \tag{3.1}$$

where $\overline{H}(x) = \lim_{n\to\infty} \mathbb{E}\left[H(x, \xi(n))\right]$.

In this perspective, our stochastic approximation algorithm can be written as

$$x(n+1) - x(n) = \gamma_n \left\{\left(-x(n) + \mathbb{E}\left[\xi(n)|\mathcal{F}_n\right]\right) + \left(\xi(n) - \mathbb{E}\left[\xi(n)|\mathcal{F}_n\right]\right)\right\}.$$

Denote $\xi(n) = (\xi_1(n), \dots, \xi_m(n))$. Observe that (2.2), (2.3) and (2.4) imply

$$\mathbb{E}\left[\xi_i(n)|\mathcal{F}_n\right] = \frac{1}{N}\sum_{j\sim i} \mathbb{E}\left[\delta_{i\leftarrow j}(n+1)|\mathcal{F}_n\right] = \frac{1}{N}\sum_{j\sim i} \frac{x_i(n)^\alpha}{x_i(n)^\alpha + x_j(n)^\alpha}\,.$$

Thus, defining $\{u_n\}_{n\geq 0} \subset \mathbb{R}^m$ by

$$u_n = \xi(n) - \mathbb{E}\left[\xi(n)|\mathcal{F}_n\right] \tag{3.2}$$

and $F = (F_1, \dots, F_m)$ to be a vector field in Δ with

$$F_i(x_1, \dots, x_m) = -x_i + \frac{1}{N}\sum_{j\sim i} \frac{x_i(n)^\alpha}{x_i(n)^\alpha + x_j(n)^\alpha}\,, \tag{3.3}$$

our random process takes the form

$$x(n+1) - x(n) = \gamma_n \left[F(x(n)) + u_n\right]. \tag{3.4}$$

The above expression is a particular case of a class of stochastic approximation algorithms studied by Benaïm in [1], on which he related the behavior of the algorithm to a weak notion of recurrence for the ODE: that of *chain-recurrence*. His theorem asserts that, under the assumptions of Kushner and Clark lemma [7], the accumulation points of $\{x(n)\}_{n\geq 0}$ are contained in the chain-recurrent set of the semiflow generated by the ODE.

In the remaining of this section, we introduce the necessary definitions for semiflows, then state Benaïm's Theorem, and conclude the section proving that our model satisfies the required conditions of his theorem.

3.1. **Preliminaries on semiflows.** Let $U \subset \mathbb{R}^m$ be an open set, and let $\Phi : \mathbb{R}_{\geq 0} \times U \to U$ be a continuous map. For simplicity, denote $\Phi(t, x)$ by $\Phi_t(x)$.

Definition 3.1 (Semiflow)**.** A *semiflow* on U is a continuous map $\Phi : \mathbb{R}_{\geq 0} \times U \to U$ such that

(i) Φ_0 is the identity on U, and
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for any $t, s \geq 0$.

To every continuous vector field $F : \mathbb{R}^m \to \mathbb{R}^m$ with unique integral curves is associated a semiflow on $\mathbb{R}^m$ by the equation

$$\frac{d}{dt}\Phi_t(x) = F(\Phi_t(x)), \quad \forall\, x \in \mathbb{R}^m, \forall\, t \in \mathbb{R}_{\geq 0}.$$

Fix a semiflow Φ on $U \subset \mathbb{R}^m$.

Definition 3.2 (Invariant set)**.** A set $\Gamma \subset U$ is called *invariant* if $\Phi_t(\Gamma) \subset \Gamma$ for every $t \geq 0$.

Definition 3.3 (Equilibrium point)**.** A point $x \in U$ is called an *equilibrium* if $\Phi_t(x) = x$ for all $t \geq 0$. The *equilibrium set* of Φ is the set of all equilibrium points.

When Φ is induced by a vector field F, the equilibrium set coincides with the set on which F vanishes.

Definition 3.4 (Chain-recurrent point)**.** Given $\delta, T > 0$, a point $x \in U$ is called (δ, T)*-recurrent* if there are points $x_0 = x, x_1, \ldots, x_{k-1}, x_k = x \in U$ and real numbers $t_0, t_1, \ldots, t_{k-1} \geq T$ such that

$$\operatorname{dist}(\Phi_{t_i}(x_i), x_{i+1}) < \delta, \quad i = 0, \ldots, k-1.$$

x is said to be *chain-recurrent* if it is (δ, T)-recurrent for any $\delta, T > 0$.

We denote by $\mathrm{CR}(\Phi)$ the set of chain-recurrent points.

Definition 3.5 (Limit set)**.** The *limit set* of a sequence $\{x_n\}_{n\geq 0} \subset U$ is the set

$$L(\{x_n\}_{n\geq 0}) = \left\{x \in \overline{U} : \exists\, n_k \uparrow \infty \text{ such that } x_{n_k} \to x\right\}.$$

3.2. **A limit set theorem.** The reason we can characterize the limit set of the random process via the chain-recurrent set of the associated semiflow is due to Theorem 1.2 of [1] which, to our purposes, is stated as

Theorem 3.6. *Let $F : \mathbb{R}^m \to \mathbb{R}^m$ be a continuous vector field with unique integral curves, and let $\{x(n)\}_{n\geq 0}$ be a solution to the recursion*

$$x(n+1) - x(n) = \gamma_n \left[F(x(n)) + u_n\right],$$

where $\{\gamma_n\}_{n\geq 0}$ is a decreasing gain sequence[1] *and $\{u_n\}_{n\geq 0} \subset \mathbb{R}^m$. Assume that*

(i) $\{x(n)\}_{n\geq 0}$ is bounded, and
(ii) for each $T > 0$,

$$\lim_{n\to\infty}\left(\sup_{\{k: 0 \leq \tau_k - \tau_n \leq T\}} \left\|\sum_{i=n}^{k-1} \gamma_i u_i\right\|\right) = 0,$$

where $\tau_n = \sum_{i=0}^{n-1} \gamma_i$.

[1] $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n\geq 0} \gamma_n = \infty$.

Then $L(\{x(n)\}_{n\geq 0})$ is contained in the chain-recurrent set of the semiflow induced by F.

Definition 3.7 (Lyapunov function)**.** Consider a semiflow Φ on $U \subset \mathbb{R}^m$. A *Lyapunov function* for Φ is a continuous map $L : W \subset U \to \mathbb{R}$ which is strictly monotone along any non-constant orbit of $\Phi|_W$. If $W = U$, we call L a *global Lyapunov function* and Φ a *gradient-like system.*

If a gradient-like system Φ is generated by a vector field F, we also say that F is gradient-like. Under these assumptions, Corollary 3.3 of [1] states that

Corollary 3.8. *Assume the conditions of Theorem 3.6, and that F is gradient-like and has isolated equilibria. Then $\{x(n)\}_{n\geq 0}$ converges toward an equilibrium.*

3.3. **The random process (3.4) satisfies Theorem 3.6.** First, note that $\{\gamma_n\}_{n\geq 0}$ satisfies

$$\lim_{n\to 0} \gamma_n = 0 \quad \text{and} \quad \sum_{n\geq 0} \gamma_n = \infty.$$

Of course, $\{x(n)\}_{n\geq 0}$ is bounded. It remains to check condition (ii). For that, let

$$M_n = \sum_{i=0}^{n} \gamma_i u_i.$$

Observe that $\{M_n\}_{n\geq 0}$ is a martingale adapted to the filtration $\{\mathcal{F}_n\}_{n\geq 0}$:

$$\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = \sum_{i=0}^{n} \gamma_i u_i + \mathbb{E}\left[\gamma_{n+1} u_{n+1}|\mathcal{F}_n\right] = \sum_{i=0}^{n} \gamma_i u_i = M_n.$$

Furthermore, because for any $n \geq 0$

$$\sum_{i=0}^{n} \mathbb{E}\left[(M_{i+1} - M_i)^2|\mathcal{F}_i\right] \leq \sum_{i=0}^{n} \gamma_{i+1}^2 \leq \sum_{i\geq 0} \gamma_i^2 < \infty \quad \text{a.s.},$$

the sequence $\{M_n\}_{n\geq 0}$ converges almost surely to a finite random variable (see e.g. Theorem 5.4.9 of [3]). In particular, it is a Cauchy sequence and so condition (ii) holds almost surely.

Now, in view of Theorem 3.6, we will investigate the chain-recurrent set of the semiflow generated by the ODE

$$\left\{\begin{array}{rcl} \dfrac{dv_1(t)}{dt} & = & -v_1(t) + \dfrac{1}{N}\displaystyle\sum_{j\sim 1} \dfrac{v_1(t)^\alpha}{v_1(t)^\alpha + v_j(t)^\alpha} \\ & \vdots & \\ \dfrac{dv_m(t)}{dt} & = & -v_m(t) + \dfrac{1}{N}\displaystyle\sum_{j\sim m} \dfrac{v_m(t)^\alpha}{v_m(t)^\alpha + v_j(t)^\alpha}\,. \end{array}\right. \tag{3.5}$$

Given $c > 0$, let

$$\Delta_c = \left\{x \in \overline{\Delta} : x_i + x_j \geq c, \forall \{i,j\} \in E\right\}.$$

On Δ_c, the vector field F of (3.5) is continuous (even Lipschitz). Moreover, we have

Lemma 3.9. *For any $c \leq 1/N$, Δ_c is invariant under the semiflow induced by (3.5).*

Proof. Given $\{i,j\} \in E$,

$$\begin{aligned}
\frac{d}{dt}(v_i+v_j) &= -v_i + \frac{1}{N}\sum_{k\sim i}\frac{v_i^\alpha}{v_i^\alpha+v_k^\alpha} - v_j + \frac{1}{N}\sum_{l\sim j}\frac{v_j^\alpha}{v_j^\alpha+v_l^\alpha} \\
&\geq -(v_i+v_j) + \frac{1}{N}\left(\frac{v_i^\alpha}{v_i^\alpha+v_j^\alpha}+\frac{v_j^\alpha}{v_j^\alpha+v_i^\alpha}\right) \\
&= -(v_i+v_j) + \frac{1}{N}\cdot
\end{aligned}$$

If v belongs to the boundary of Δ_c, there exists some $\{i,j\} \in E$ such that $v_i+v_j = c$, and then

$$\frac{d}{dt}(v_i+v_j) \geq -(v_i+v_j) + \frac{1}{N} = -c + \frac{1}{N} \geq 0,$$

which means that the vector field F points inward on the boundary of Δ_c. □

The proofs of Theorem 1.1 and Theorem 1.2 will follow from the description of the chain-recurrent set of (3.5).

4. The case $\alpha = 1$ for regular graphs: proof of Theorem 1.1

Here, $\alpha = 1$ and G is a r-regular, finite, connected graph. According to Theorem 3.6, the limit set of $\{x(n)\}_{n\geq 0}$ is contained in the chain-recurrent set $\mathrm{CR}(\Phi)$ of the semiflow Φ. Therefore it is enough to characterize $\mathrm{CR}(\Phi)$. We do this by introducing a global Lyapunov function for Φ. Let $L : \Delta \to \mathbb{R}$ be given by

$$L(v_1,\ldots,v_m) = \log\prod_{i=1}^m v_i = \sum_{i=1}^m \log v_i. \tag{4.1}$$

Lemma 4.1. *$L : \Delta \to \mathbb{R}$ is a global Lyapunov function for the semiflow induced by (3.5).*

Proof. Let $v = (v_1(t),\ldots,v_m(t))$, $t \geq 0$, be an orbit of Φ. Then

$$\begin{aligned}
\frac{d}{dt}(L\circ v) &= \frac{d}{dt}\left(\sum_{i=1}^m \log v_i\right) \\
&= \sum_{i=1}^m \frac{1}{v_i}\cdot\frac{dv_i}{dt} \\
&= \sum_{i=1}^m \frac{1}{v_i}\left(-v_i + \frac{1}{N}\sum_{j\sim i}\frac{v_i}{v_i+v_j}\right) \\
&= -m + \frac{1}{N}\sum_{i=1}^m\sum_{j\sim i}\frac{1}{v_i+v_j}\cdot
\end{aligned}$$

We claim that the last expression is nonnegative. For this, note that the above summand has $2N$ terms and, by the arithmetic-harmonic mean inequality,

$$\left[\sum_{i=1}^m\sum_{j\sim i}\frac{1}{v_i+v_j}\right]\cdot\left[\sum_{i=1}^m\sum_{j\sim i}(v_i+v_j)\right] \geq (2N)^2, \tag{4.2}$$

with equality if and only if

$$v_i + v_j = \text{const.}\ , \quad \forall \{i,j\} \in E. \tag{4.3}$$

Since G is r-regular, we have $N = rm/2$ and

$$\sum_{i=1}^{m} \sum_{j \sim i} (v_i + v_j) = 2r.$$

So (4.2) gives that

$$\sum_{i=1}^{m} \sum_{j \sim i} \frac{1}{v_i + v_j} \geq \frac{(2N)^2}{2r} = Nm, \tag{4.4}$$

thus establishing that $\frac{d}{dt}(L \circ v) \geq 0$. It remains to prove that equality only holds along constant orbits. By (4.3), we must have $v_i + v_j = \text{const.} = r/N$ for every $\{i,j\} \in E$. Thus

$$\frac{dv_i}{dt} = -v_i + \frac{1}{N} \sum_{j \sim i} \frac{v_i}{v_i + v_j} = -v_i + \frac{1}{N} \sum_{j \sim i} \frac{v_i}{r/N} = 0$$

and so v is an equilibrium point. □

The proof of Theorem 1.1 is divided into two cases.

4.1. **Case 1: G is non-bipartite.** By the existence of a global Lyapunov function, every orbit $\{v(t)\}_{t \geq 0}$ converges to an equilibrium point. We claim that $u = (1/m, \ldots, 1/m)$ is the only equilibrium of (3.5). For this, note that every equilibrium satisfies

$$\frac{d}{dt}(L \circ v) = 0.$$

By the proof of Lemma 4.1, this happens if and only if (4.3) is satisfied. Fix the vertex 1 of G, and let $v_1 = p$ and $v_i = q$ for every neighbor $i \sim 1$. By (4.3), the values of v_i along any path in the graph G alternate between p and q, i.e.

$$v_i \quad = \quad \begin{cases} p & \text{if the distance from } i \text{ to } 1 \text{ is even,} \\ q & \text{if the distance from } i \text{ to } 1 \text{ is odd.} \end{cases}$$

Because G is connected and non-bipartite, it has a cycle of odd length, and so $p = q$. This implies that $v = u$. It is easy to check that u is indeed an equilibrium point of (3.5).

We are thus in the conditions of Corollary 3.8, and so $\{x(n)\}_{n \geq 0}$ converges to u. This proves the first part of Theorem 1.1.

4.2. **Case 2: G is bipartite.** The situation here is quite different from the previous one. Let $V = A \cup B$ be a bipartition of G. Because G is a regular graph, A and B have the same cardinality. Observe that, by the same argument of the previous section, if $v = (v_1, \ldots, v_m)$ is an equilibrium point, there exist $p, q > 0$ such that

$$v_i \quad = \quad \begin{cases} p & \text{if } i \in A, \\ q & \text{if } i \in B. \end{cases} \tag{4.5}$$

Because $v \in \Delta$, we have $p + q = 2/m$. Thus every equilibrium point of (3.5) is contained in the set

$$\Lambda = \{v \in \Delta : v \text{ satisfies } (4.5)\}.$$

Observe that every $v \in \Lambda$ is indeed an equilibrium point: for $i \in A$

$$F_i(v) = -v_i + \frac{1}{N} \sum_{j \sim i} \frac{v_i}{v_i + v_j} = -p + \frac{rp}{N(p+q)} = 0,$$

and the same holds for $i \in B$. Thus Λ is the equilibrium set of (3.5).

The goal is to prove that $\mathrm{CR}\,(\Phi) = \Lambda$. This is not guaranteed by the fact that each orbit converges toward an element of Λ. One could have, for example, a homoclinic connection: an orbit $\{v(t)\}_{t \geq 0}$ is born and dies in points of Λ. In this case, the whole orbit $\{v(t)\}_{t \geq 0}$ is contained in $\mathrm{CR}\,(\Phi)$. To avoid such situations, we want to prove that Λ is an attractor.

Definition 4.2 (Attractor)**.** Γ is an *attractor* for Φ if

(i) Γ is a compact, nonempty, invariant set for Φ, and
(ii) there exists a neighborhood W of Γ such that $\mathrm{dist}\,(\Phi_t(x), \Gamma) \to 0$ as $t \to \infty$, uniformly in $x \in W$.

This is not quite true, but almost. What happens is that elements of Λ are attracting equilibria points, but as they approach the two "points at infinity" of Λ (i.e. when p or q goes to zero), some of the eigenvalues of JF on complementary directions of Λ converge to zero, which means that they lose hyperbolicity. We bypass this problem by extending the semiflow to neighborhoods of these "points at infinity" and showing that the closure $\overline{\Lambda}$ is an attractor for the extended semiflow.

We first prove that every $v \in \Lambda$ is an attracting equilibrium, by looking at the jacobian matrix

$$JF = \begin{bmatrix} \dfrac{\partial F_1}{\partial v_1} & \cdots & \dfrac{\partial F_1}{\partial v_m} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial v_1} & \cdots & \dfrac{\partial F_m}{\partial v_m} \end{bmatrix} \tag{4.6}$$

of the vector field $F = (F_1, \ldots, F_m)$ defined by (3.5). Because $v \in \Lambda$ belongs to a line of singularities, 0 is an eigenvalue of $JF(v)$. To be attracting, we must show that

Lemma 4.3. *Let $v \in \Lambda$. Any eigenvalue of $JF(v)$ different from 0 has negative real part, and 0 is a simple eigenvalue of $JF(v)$.*

Proof. We explicitly calculate the entries $\partial F_i / \partial v_k$. Let $v \in \Lambda$ with $v_i = p$ for $i \in A$ and $v_i = q$ for $i \in B$. We have five cases:

- $i = k \in A$:

$$\frac{\partial F_i}{\partial v_i}(v) = -1 + \frac{1}{N} \sum_{j \sim i} \frac{v_j}{(v_i + v_j)^2} = -1 + \frac{mq}{2} \cdot$$

- $i = k \in B$: analogously,

$$\frac{\partial F_i}{\partial v_i}(v) = -1 + \frac{mp}{2} \cdot$$

- $i \sim k$ and $i \in A$:

$$\frac{\partial F_i}{\partial v_k}(v) = -\frac{v_i}{N(v_i + v_k)^2} = -\frac{mp}{2r} \cdot$$

• $i \sim k$ and $i \in B$: analogously,

$$\frac{\partial F_i}{\partial v_k}(v) = -\frac{mq}{2r}\cdot$$

• $i \not\sim k$: in this case, $\partial F_i/\partial v_k = 0$.

Thus, if we label the vertices of A from 1 to $m/2$, the vertices of B from 1 to $m/2$, and let $M = (m_{ij})$ be the $m/2 \times m/2$ adjacency matrix of the edges connecting vertices of A to vertices of B (i.e. $m_{ij} = 1$ when the i-th vertex of A is adjacent to the j-th vertex of B), then $JF(v)$ takes the form

$$JF(v) = -I + \frac{m}{2r}\begin{bmatrix} rqI & -pM \\ & \\ -qM^t & rpI \end{bmatrix}.$$

Letting $\mu = p/(p+q)$ and $\nu = q/(p+q)$, $JF(v)$ can be written as

$$JF(v) = -I + \frac{1}{r}\begin{bmatrix} r\nu I & -\mu M \\ & \\ -\nu M^t & r\mu I \end{bmatrix} =: -I + \frac{1}{r}S. \tag{4.7}$$

Given a matrix X, let $\sigma(X)$ denote its spectrum. By (4.7),

$$\sigma(JF(v)) = \frac{1}{r}\sigma(S) - 1$$

and so it is enough to estimate the set $\sigma(S)$. The lemma will follow once we prove that

(a) every element of $\sigma(S)$ is either real or has real part equal to $r/2$,
(b) r is the largest real eigenvalue of S, and
(c) r is a simple eigenvalue of S.

Let's prove (a). Let $\lambda = a + bi \in \sigma(S)$. Because $r\mu, r\nu < r$, we can assume that $\lambda \neq r\mu, r\nu$. Note that the matrix

$$S - \lambda I = \begin{bmatrix} (r\nu - \lambda)I & -\mu M \\ & \\ -\nu M^t & (r\mu - \lambda)I \end{bmatrix}$$

is singular if and only if its Schur complement

$$(r\nu - \lambda)I - (-\mu M)(r\mu - \lambda)^{-1}I(-\nu M^t) = \frac{\mu\nu}{r\mu - \lambda}\left[\frac{(r\mu - \lambda)(r\nu - \lambda)}{\mu\nu}I - MM^t\right]$$

is singular. Because MM^t is symmetric, all of its eigenvalues are real and so

$$\frac{(r\mu - \lambda)(r\nu - \lambda)}{\mu\nu} \in \mathbb{R} \Longrightarrow (r\mu - \lambda)(r\nu - \lambda) \in \mathbb{R}.$$

Calculating the imaginary part of $(r\mu - \lambda)(r\nu - \lambda)$, it follows that

$$-rb + 2ab = 0 \Longrightarrow b = 0 \text{ or } a = r/2,$$

which proves (a).

To prove (b), let $\lambda \in \sigma(S) \cap \mathbb{R}$, say $Sx = \lambda x$, where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m \backslash \{0\}$. Letting $x_i = \max\{|x_1|, \ldots, |x_m|\}$ for $i \in A$, we have

$$\lambda x_i = r\nu x_i - \mu \sum_{j \sim i} x_j \leq r\nu x_i + \mu \sum_{j \sim i} x_i = r x_i$$

and thus $\lambda \le r$. The same holds if $i \in B$.

It remains to prove (c). When $\lambda = r$, we have for $i \in A$ that

$$rx_i = r\nu x_i - \mu \sum_{j\sim i} x_j \Longrightarrow x_i = -\frac{1}{r}\sum_{j\sim i} x_j$$

and similarly for $i \in B$. Thus the function $h : V \to \mathbb{R}$ defined by

$$h(i) = \begin{cases} x_i & \text{if } i \in A, \\ -x_i & \text{if } i \in B \end{cases}$$

is harmonic in G. By the maximum principle, h is constant, and so r is a simple eigenvalue of S. □

Let $\{v_{-\infty}, v_\infty\} = \overline{\Lambda}\backslash\Lambda \subset \partial\Delta$ be the points at infinity of Λ, that is

$$v_{-\infty,i} = \begin{cases} 0 & \text{if } i \in A, \\ \frac{2}{m} & \text{if } i \in B \end{cases} \qquad \text{and} \qquad v_{\infty,i} = \begin{cases} \frac{2}{m} & \text{if } i \in A, \\ 0 & \text{if } i \in B. \end{cases}$$

Observe that, because $v_{-\infty,i} + v_{-\infty,j}, v_{\infty,i} + v_{\infty,j} > 0$ whenever $i \sim j$, there are neighborhoods $U_{-\infty} \subset \overline{\Delta}$ of $v_{-\infty}$ and $U_\infty \subset \overline{\Delta}$ of v_∞ on which the vector field F can be extended to a vector field $\overline{F}$ on $\Delta \cup U_{-\infty} \cup U_\infty$. Let $\overline{\Phi}$ be the semiflow induced by $\overline{F}$. Observe that $v_{-\infty}$ and v_∞ are equilibria of $\overline{\Phi}$. But now

$$JF(v_{-\infty}) = -I + \frac{1}{r}\begin{bmatrix} rI & 0 \\ & \\ -M^t & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ & \\ -\frac{1}{r}M^t & -I \end{bmatrix}$$

has $m/2$ eigenvalues equal to 0, and the same happens to $JF(v_\infty)$. In particular, the analysis of eigenvalues does not give any information about the local behavior of $\overline{\Phi}$ around these points. We overcome this difficulty by introducing new Lyapunov functions on neighborhoods of $v_{-\infty}$ and v_∞. Let $L_{-\infty} : U_{-\infty} \to \mathbb{R}$ and $L_\infty : U_\infty \to \mathbb{R}$ be defined by

$$L_{-\infty}(v) = \sum_{i\in B} \log v_i \qquad \text{and} \qquad L_\infty(v) = \sum_{i\in A} \log v_i.$$

Observe that $v_{-\infty}$ is a global maximum of $L_{-\infty}$: by the concavity of the log function, Jensen's inequality gives that for every $v \in U_{-\infty}$

$$L_{-\infty}(v) \le \frac{m}{2}\log\left(\frac{2}{m}\sum_{i\in B} v_i\right) \le \frac{m}{2}\log\left(\frac{2}{m}\right) = L_{-\infty}(v_{-\infty}).$$

Analogously, v_∞ is a global maximum of L_∞.

Lemma 4.4. *$L_{-\infty}$ and L_∞ are Lyapunov functions for the semiflow $\overline{\Phi}$.*

Proof. We only prove the first claim (the second claim is analogous). Like in the proof of Lemma 4.1,

$$\frac{d}{dt}(L_{-\infty} \circ v) = -\frac{m}{2} + \frac{1}{N}\sum_{i\in B}\sum_{j\sim i}\frac{1}{v_i + v_j}$$

which, again by the arithmetic-harmonic mean inequality, is nonnegative. Also, it is zero if and only if $v \in \overline{\Lambda} \cap U_{-\infty}$. □

The existence of the functions $L, L_{-\infty}, L_\infty$ guarantees, in particular, that every orbit of $\overline{\Phi}$ converges to a point of $\overline{\Lambda}$.

We now proceed to establish the following

Lemma 4.5. *$\overline{\Lambda}$ is an attractor for the semiflow $\overline{\Phi}$.*

Proof. Given $\varepsilon_0 > 0$, let

$$\begin{aligned} X &= \{x \in U_{-\infty} : L_{-\infty}(x) \geq L_{-\infty}(v_{-\infty}) - \varepsilon_0\} \\ Y &= \{y \in U_{\infty} : L_{\infty}(y) \geq L_{\infty}(v_{\infty}) - \varepsilon_0\}. \end{aligned}$$

These are compact sets containing $v_{-\infty}$ and v_∞, respectively. Fix $x \in \mathrm{int}(X) \cap \Lambda$ and $y \in \mathrm{int}(Y) \cap \Lambda$. They define an interval I of Λ. By Lemma 4.3, we can take an arbitrarily small tubular neighborhood Z of I that is invariant under Φ. Thus, by Lemmas 4.3 and 4.4, the union $W = X \cup Y \cup Z$ is invariant under $\overline{\Phi}$.

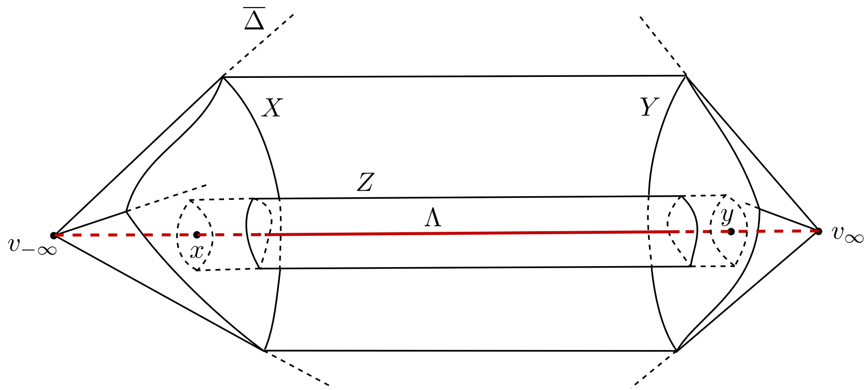

FIGURE 1. The attracting neighborhood $X \cup Y \cup Z$ of $\overline{\Lambda}$.

It remains to check condition (ii) of Definition 4.2. Observe that because $v_{-\infty}, v_\infty$ are global maxima for $L_{-\infty}, L_\infty$ respectively and each point of Λ is uniformly attracting, for every $\varepsilon > 0$ we can similarly build a smaller invariant set $W' = X' \cup Y' \cup Z'$ contained in a ε-neighborhood of $\overline{\Lambda}$. Now, since W is compact and every point of W converges to a point of $\overline{\Lambda}$, there is $t_0 > 0$ such that $\Phi_t(W) \subset W'$ for any $t > t_0$. $\square$

Now it is an easy task to show that

Lemma 4.6. $\mathrm{CR}\left(\overline{\Phi}\right) = \overline{\Lambda}$.

Proof. Obviously, $\overline{\Lambda} \subset \mathrm{CR}\left(\overline{\Phi}\right)$. For the reverse inclusion, assume by contradiction that $z \notin \overline{\Lambda}$ is chain-recurrent. By the previous lemma, we can take a $\overline{\Phi}$-invariant compact set W containing $\overline{\Lambda}$ that does not contain z. By construction, $W' = \overline{\Phi}_1(W)$ is also $\overline{\Phi}$-invariant, compact and satisfies

$$\overline{\Lambda} \subset W' \subset \mathrm{int}(W). \tag{4.8}$$

Let $\delta = \mathrm{dist}\left(W', \overline{\Delta} \backslash \mathrm{int}(W)\right) > 0$. By assumption, there are points $z_0 = z, z_1, \ldots, z_{k-1}, z_k = z \in \overline{\Delta}$ and real numbers $t_0, \ldots, t_{k-1} > 1$ such that

$$\mathrm{dist}\left(\overline{\Phi}_{t_i}(z_i), z_{i+1}\right) < \delta, \quad i = 0, \ldots, k-1. \tag{4.9}$$

Thus $\Phi_{t_0}(z_0) = \Phi_{t_0}(z) \in W'$ and so, by (4.9), $z_1 \in W$. By induction, we claim that $z_1, z_2, \ldots, z_k \in W$. Indeed, if $z_i \in W$ then $\Phi_{t_i}(z_i) \in W'$, and again by (4.9), $z_{i+1} \in W$. In particular, $z_k = z \in W$, which contradicts the choice of W. This concludes the proof. $\square$

5. The case $\alpha < 1$ for regular graphs: proof of Theorem 1.2

We claim that the same L, as defined in (4.1), is a global Lyapunov function whenever $\alpha < 1$.

Lemma 5.1. *For any $\alpha < 1$, $L : \Delta \to \mathbb{R}$ is a global Lyapunov function for the semiflow induced by (3.5).*

Proof. Let $v = (v_1(t), \ldots, v_m(t))$, $t \geq 0$, be an orbit of Φ. Then

$$\begin{aligned}\frac{d}{dt}(L \circ v) &= \frac{d}{dt}\left(\sum_{i=1}^{m} \log v_i\right) \\ &= \sum_{i=1}^{m} \frac{1}{v_i} \cdot \frac{dv_i}{dt} \\ &= \sum_{i=1}^{m} \frac{1}{v_i}\left(-v_i + \frac{1}{N}\sum_{j\sim i} \frac{v_i^\alpha}{v_i^\alpha + v_j^\alpha}\right) \\ &= -m + \frac{1}{N}\sum_{\{i,j\}\in E} \frac{v_i^{\alpha-1} + v_j^{\alpha-1}}{v_i^\alpha + v_j^\alpha}.\end{aligned}$$

Observe that[2]

$$\frac{x^{\alpha-1} + y^{\alpha-1}}{x^\alpha + y^\alpha} \geq \frac{2}{x+y} \quad \text{for any } x, y > 0, \tag{5.1}$$

with equality if and only if $x = y$. Thus

$$\frac{d}{dt}(L \circ v) = -m + \frac{1}{N}\sum_{\{i,j\}\in E} \frac{v_i^{\alpha-1} + v_j^{\alpha-1}}{v_i^\alpha + v_j^\alpha} \geq -m + \frac{1}{N}\sum_{\{i,j\}\in E} \frac{2}{v_i + v_j}\,, \tag{5.2}$$

and this last expression, by inequality (4.4), is nonnegative.

It remains to check at which points $\frac{d}{dt}(L \circ v) = 0$. This happens if and only if equality holds in both (5.2) and (4.4). Equality holds in (4.4) if and only if

$$v_i + v_j = \text{const.}\,, \quad \forall\, \{i,j\} \in E, \tag{5.3}$$

and in (5.2) if and only if

$$v_i = v_j\,, \quad \forall\, \{i,j\} \in E. \tag{5.4}$$

Clearly, (5.3) and (5.4) imply that v is the uniform measure. It is easy to check that the uniform measure is an equilibrium point. □

Thus the semiflow is gradient-like and the uniform measure is the unique equilibrium point. By Corollary 3.8, $x(n)$ converges to the uniform measure almost surely.

[2]The inequality is equivalent to $(x^{\alpha-1} - y^{\alpha-1})(x - y) \leq 0$, which is true for $\alpha < 1$.

6. The case of star graphs: proof of Theorem 1.3

When G is the star graph with m vertices and m is the vertex with higher degree, (3.5) becomes

$$\begin{cases} \dfrac{dv_i(t)}{dt} = -v_i(t) + \dfrac{1}{m-1} \cdot \dfrac{v_i(t)^\alpha}{v_i(t)^\alpha + v_m(t)^\alpha}, & i = 1, \ldots, m-1, \\ \dfrac{dv_m(t)}{dt} = -v_m(t) + \dfrac{1}{m-1} \displaystyle\sum_{j=1}^{m-1} \dfrac{v_m(t)^\alpha}{v_m(t)^\alpha + v_j(t)^\alpha}. \end{cases} \tag{6.1}$$

6.1. **Case 1: $\alpha = 1$.** Let $L : \Delta \to \mathbb{R}$ be given by

$$L(v_1, \ldots, v_m) = \log v_m. \tag{6.2}$$

Lemma 6.1. *$L : \Delta \to \mathbb{R}$ is a global Lyapunov function for the semiflow induced by (6.1).*

Proof. Similar to the proof of Lemma 4.1,

$$\frac{d}{dt}(L \circ v) = -1 + \frac{1}{m-1} \sum_{j=1}^{m-1} \frac{1}{v_m + v_j}$$

which, by the arithmetic-harmonic mean inequality, is at least

$$-1 + \frac{m-1}{\displaystyle\sum_{j=1}^{m-1}(v_m + v_j)} = -1 + \frac{m-1}{1 + (m-2)v_m} \geq -1 + \frac{m-1}{1 + (m-2)} = 0,$$

with equality if and only if $v_1 = \cdots = v_{m-1} = 0$ and $v_m = 1$. It is easy to check that $(0, \ldots, 0, 1)$ is indeed an equilibrium point of (6.1). □

By Corollary 3.8, every orbit of Φ converges to $(0, \ldots, 0, 1)$.

6.2. **Case 2: $\alpha > 1$.** This is the only result we make no use of Lyapunov functions. When $m = 2$, our model is a class of generalized Pólya's urn. For simplicity, we refer to this process as "g-urn". It is known (see e.g. Theorem 4.1 of [4]) that in this case

$$\mathbb{P}\left[\lim_{n\to\infty} x(n) = (0,1)\right] > 0 \quad \text{and} \quad \mathbb{P}\left[\lim_{n\to\infty} x(n) = (1,0)\right] > 0.$$

Now assume $m > 2$. Observe that, as events,

$$\left\{\lim_{n\to\infty} x(n) = \left(\frac{1}{m-1}, \ldots, \frac{1}{m-1}, 0\right)\right\} \supset \bigcap_{i=1}^{m-1} \bigcap_{n\geq 1} \{\delta_{i\leftarrow m}(n) = 1\}.$$

By a coupling argument, we can identify this last event to the following one: in $m-1$ independent g-urns, just one color of ball is added in each g-urn since the beginning of the process. Rubin's Theorem (see e.g. Theorem 3.6 of [12]) guarantees that the event "just one color of ball is added to the g-urn since the beginning of the process" has positive probability, and so

$$\mathbb{P}\left[\lim_{n\to\infty} x(n) = \left(\frac{1}{m-1}, \ldots, \frac{1}{m-1}, 0\right)\right] > 0.$$

To prove the other claim, first observe that

$$\left\{\lim_{n\to\infty} x(n) = (0,\dots,0,1)\right\} \supset \bigcap_{i=1}^{m-1} \bigcap_{n\geq 1} \{\delta_{i\leftarrow m}(n) = 0\}.$$

By a coupling argument, the term on the right hand side of the above inclusion has positive probability (again by Rubin's Theorem). This concludes the proof of Theorem 1.3.

Remark 6.2. Given a finite connected graph $G = (V, E)$, call $I \subset V$ an *independent set* if $\{i, j\} \notin E$ for $i, j \in I$. The proof of Theorem 1.3 gives the following: if $\alpha > 1$ and I is an independent set, then

$$\mathbb{P}\left[\lim_{n\to\infty} x_i(n) = 0,\ \forall\, i \in I\right] > 0.$$

7. Variants of the model

7.1. **Edges with different weight functions.** Let $G = (V, E)$ be a finite, regular, connected graph. For each edge $\{i, j\} \in E$, let $f_{\{i,j\}} : (0, 1) \to (0, 1)$. A variant of the model is described as follows. Let $x_1(n-1), \dots, x_m(n-1)$ be the proportions of balls after step $n-1$. At step n, for each edge $\{i, j\} \in E$ add one ball either to i or j with probability

$$\mathbb{P}\left[i \text{ is chosen among } \{i,j\} \text{ at step } n\right] = \frac{f_{\{i,j\}}(x_i(n-1))}{f_{\{i,j\}}(x_i(n-1)) + f_{\{i,j\}}(x_j(n-1))}\,.$$

In other words, we replace the law of $\delta_{i\leftarrow j}(n)$ in (2.2) by the above one, defined in terms of the $f_{\{i,j\}}$'s.

If we assume that, for each $\{i, j\} \in E$, the function $x \mapsto f_{\{i,j\}}(x)/x$ is decreasing, an inequality similar to (5.1) holds:

$$\frac{f_{\{i,j\}}(x)/x + f_{\{i,j\}}(y)/y}{f_{\{i,j\}}(x) + f_{\{i,j\}}(y)} \geq \frac{2}{x+y} \quad \text{for any } x, y > 0,$$

with equality if and only if $x = y$. This implies that the same function L defined in (4.1) is a Lyapunov function for this variant model. Following the same line of ideas in the proof of Theorem 1.2, we conclude that $x(n)$ converges to the uniform measure almost surely. A special family of functions satisfying the above assumption is $f_{\{i,j\}}(x) = x^\alpha$, where $\alpha = \alpha(\{i, j\}) < 1$.

7.2. **Hypergraph based interactions.** We can similarly define a variant of the model on hypergraphs. Let $G = (V, E)$ be an hypergraph, where $V = \{1, \dots, m\}$ and $E \subset 2^V$, $|E| = N$. Let $x_1(n-1), \dots, x_m(n-1)$ be the proportions of balls after step $n-1$. At step n, for each hyperedge $e \in E$ add one ball to one of its vertices with probability

$$\mathbb{P}\left[i \text{ is chosen on hyperedge } e \text{ at step } n\right] = \frac{x_i(n-1)}{\sum_{j\in e} x_j(n-1)}\,.$$

The associated ODE is

$$\left\{\begin{array}{rcl} \dfrac{dv_1(t)}{dt} & = & -v_1(t) + \dfrac{1}{N}\displaystyle\sum_{\substack{e\in E\\ 1\in e}} \dfrac{v_1(t)}{\sum_{j\in e} v_j(t)}\\ & \vdots & \\ \dfrac{dv_m(t)}{dt} & = & -v_m(t) + \dfrac{1}{N}\displaystyle\sum_{\substack{e\in E\\ m\in e}} \dfrac{v_m(t)}{\sum_{j\in e} v_j(t)}\,. \end{array}\right. \tag{7.1}$$

If G is

(i) regular: each $i \in V$ belongs to a same number of hyperedges, and
(ii) k-uniform: $|e| = k$ for every $e \in E$,

then L as in (4.1) is a Lyapunov function for (7.1), by the same arguments used in the proof of Lemma 4.1. Thus every orbit $\{v(t)\}_{t\geq 0}$ converges toward an equilibrium point. An equilibrium point $v = (v_1, \dots, v_m)$ must satisfy the system of equations

$$\sum_{j\in e} v_j = \text{const.}\,, \quad \forall\, e \in E. \tag{7.2}$$

As a special case, let $E = \{e \in 2^V : |e| = k\}$. When $k < m$, $u = (1/m, \dots, 1/m)$ is the only solution to the system of equations (7.2). Thus $x(n)$ converges to the uniform measure almost surely. When $k = m$, this variant is a Pólya's urn model with balls of m colors. See for instance §4.2 of [12].

In addition to the above two variants of the model, one can also consider a variant in which the number of balls added at each step is some process possibly depending on the outcome so far.

8. Further questions

This work is part of a program to answer the following

Problem 8.1. *Given $\alpha > 0$ and a finite connected graph G, what is the limiting behavior of $x(n)$?*

Now turn attention to the cases we considered. We summarize them in the table below.

<table>
<tr><th>G</th><th>Regular non-bipartite</th><th>Regular bipartite</th><th>Star graph</th></tr>
<tr><td>$\alpha < 1$</td><td rowspan="2">uniform measure</td><td>uniform measure</td><td>?</td></tr>
<tr><td>$\alpha = 1$</td><td>$\overline{\Lambda}$</td><td>$(0, \dots, 0, 1)$</td></tr>
<tr><td>$\alpha > 1$</td><td>?</td><td>?</td><td>Probability > 0 to $(0, \dots, 0, 1)$ and $\left(\frac{1}{m-1}, \dots, \frac{1}{m-1}, 0\right)$</td></tr>
</table>

Table 1. The limiting behavior of $x(n)$.

When G is regular bipartite and $\alpha = 1$, Theorem 1.1 says that $L(\{x(n)\}_{n\geq 0}) \subset \overline{\Lambda}$. However, we do not know if the limit exists. When the number of vertices is two,

the model is the classical Pólya's urn, and in this case a well known fact is that $\{x(n)\}_{n\geq 0}$ converges to a point of $\overline{\Lambda}$ almost surely. See e.g. §2.1 of [12].

Problem 8.2. *For a general regular bipartite graph and $\alpha = 1$, does $\{x(n)\}_{n\geq 0}$ converge to a point of $\overline{\Lambda}$ almost surely?*

Problem 8.3. *In Theorems 1.1 and 1.2, what is the rate of convergence of $x(n)$ to its limit?*

This problem is related to the control of the eigenvalues of JF at the singularities of F, and of quantitative estimates on the precision that $\{x(n)\}_{n\geq 0}$ shadows a real orbit of Φ. See e.g. §3.2 of [12]. For regular non-bipartite graphs and $\alpha = 1$, direct calculations show that every eigenvalue of JF at the uniform measure has negative real part.

Problem 8.4. *Let G be a finite connected graph, and let $\alpha > 1$. What is the asymptotic behavior of $x(n)$?*

This question remains widely open, even when G is a triangle. The uniform measure is always a singularity of Φ. When $1 < \alpha < 4/3$, it is attracting and then $x(n)$ converges to it with positive probability. Also, by Remark 6.2, for any $i \in \{1, 2, 3\}$, $x_i(n)$ converges to zero with positive probability. So in general, we think that there exists $\alpha_0 = \alpha_0(G)$ such that when $\alpha > \alpha_0$

$$\mathbb{P}\left[\lim_{n\to\infty} x(n) \in \partial\Delta\right] = 1.$$

Problem 8.5. *Let G be a finite star graph with at least three vertices, and let $\alpha > 1$. What is the asymptotic behavior of $x(n)$?*

Item (b) of Theorem 1.3 gives a partial answer. We think that $x(n)$ almost surely converges to either $(0, \ldots, 0, 1)$ or $(1/(m-1), \ldots, 1/(m-1), 0)$.

Problem 8.6. *Let G be a finite star graph with at least three vertices, and let $\alpha < 1$. What is the asymptotic behavior of $x(n)$?*

When $\alpha = 0$, the probability of adding a ball to each extreme of an edge is $1/2$. By the law of large numbers,

$$\lim_{n\to\infty} x(n) = \left(\frac{1}{2(m-1)}, \ldots, \frac{1}{2(m-1)}, \frac{1}{2}\right) \quad \text{a.s.}$$

When $\alpha = 1$, item (a) of Theorem 1.3 says $x(n)$ converges to $(0, \ldots, 0, 1)$ almost surely. For $\alpha < 1$, we think that $x(n)$ almost surely converges to

$$\left(\frac{1}{m-1+(m-1)^{\frac{1}{1-\alpha}}}, \ldots, \frac{1}{m-1+(m-1)^{\frac{1}{1-\alpha}}}, \frac{(m-1)^{\frac{1}{1-\alpha}}}{m-1+(m-1)^{\frac{1}{1-\alpha}}}\right).$$

A direct calculation shows that the above point is the unique equilibrium of (3.5) inside Δ.

Another question of interest is the following

Problem 8.7. *What is the correlation between the number of balls in the bins, as a function of α and of the number of steps n, e.g. when G is an Euclidean lattice?*

9. Acknowledgements

The authors are thankful to Omer Angel and Omri Sarig for valuable comments and suggestions. I.B. is the incumbent of the Renee and Jay Weiss Professorial Chair. J.C. is supported by the ISF. Y.L. is supported by the ERC, grant 239885.

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